

Homework 1

1. **Estimating $(1 - x)$ using $\exp(\cdot)$ function.** For $x \in [0, 1)$, we know that

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

- (a) **(5 points)** Prove that $1 - x \leq \exp\left(-x - \frac{x^2}{2}\right)$.

Solution.

(b) **(5 points)** For $x \in [0, 1/2]$, prove that

$$1 - x \geq \exp(-x - x^2).$$

Solution.

2. **Tight Estimations** Provide meaningful upper and lower bounds for the following expressions.

(a) **(5 points)** $S = \sum_{i=1}^{\infty} i^{-\frac{17}{15}}$.

Note: Please evaluate/simplify the expression/bound as much as possible.

Hint: Your upper and lower bounds should be constants.

Solution.

(b) **(10 points)** $A_n = {}_{2n}P_n$ Hint: Note that ${}_{2n}P_n = \frac{(2n)!}{(2n-n)!}$.

Note: Please evaluate/simplify the expression/bound as much as possible.

Hint: You may want to start by upper and lower bounding $S_n = \sum_{i=1}^n \ln i$.

Solution.

3. **Understanding Joint Distribution.** Ten balls are to be tossed into five bins numbered $\{1, 2, 3, 4, 5\}$. Each ball is thrown into a bin uniformly and independently into the bins. For $i \in \{1, 2, 3, 4, 5\}$, let X_i represent the number of balls that fall into bin i .

- (a) **(5 points)** Find the (marginal) distribution of X_5 and compute its expected value.

Solution.

- (b) **(3 points)** Find the expected value of $X_1 + X_3 + X_5$.

Solution.

(c) **(7 points)** Find $\mathbb{P}[X_2 = 3 | X_1 + X_3 + X_5 = 6]$.

Solution.

4. Sending one bit.

Alice intends to send a bit $b \in \{0, 1\}$ to Bob. When Alice sends the bit, it goes through a series of n relays before reaching Bob. Each relay flips the received bit independently with probability p before forwarding that bit to the next relay.

- (a) **(5 points)** Show that Bob will receive the correct bit with probability

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k} \cdot (1-p)^{n-2k}.$$

Hint: Be careful that Alice could be sending either 0 or 1.

Solution.

- (b) **(5 points)** Let us consider an alternative way to calculate this probability. We say that the relay has *bias* q if the probability it flips the bit is $(1 - q)/2$. The bias q is a real number between -1 and $+1$. Show that sending a bit through two relays with bias q_1 and q_2 is equivalent to sending a bit through a single relay with bias $q_1 \cdot q_2$.

Solution.

- (c) **(5 points)** Prove that the probability you receive the correct bit when it passes through n relays is

$$\frac{1 + (1 - 2p)^n}{2}.$$

Solution.

5. An Useful Estimate.

For an integers n and t satisfying $0 \leq t \leq n/2$, define

$$P_n(t) = \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{t}{n}\right)$$

We will estimate the above expression. (*Remark:* You shall see the usefulness of this estimation in the topic “Birthday Bound” that we shall cover in the forthcoming lectures.)

(a) **(13 points)** Show that

$$\exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta(t^3)}{6n^2}\right) \geq P_n(t) \geq \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta(t^3)}{3n^2}\right).$$

Solution.

- (b) **(2 points)** Show that when $t = \sqrt{2cn}$, where c is a positive constant, the expression above is

$$P_n(t) = \exp\left(-c - \Theta(1/\sqrt{n})\right).$$

Solution.